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Classical limit of non-Hermitian quantum dynamics—a generalized canonical structure

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Abstract
We investigate the classical limit of non-Hermitian quantum dynamics arising from a coherent state approximation, and show that the resulting classical phase space dynamics can be described by generalized 'canonical' equations of motion, for both conservative and dissipative motion. The dynamical equations combine a symplectic flow associated with the Hermitian part of the Hamiltonian with a metric gradient flow associated with the anti-Hermitian part of the Hamiltonian. We derive this structure of the classical limit of quantum systems in the case of a Euclidean phase space geometry. As an example we show that the classical dynamics of a damped and driven oscillator can be linked to a non-Hermitian quantum system, and investigate the quantum classical correspondence. Furthermore, we present an example of an angular momentum system whose classical phase space is spherical and show that the generalized canonical structure persists for this nontrivial phase space geometry.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

While quantum mechanics traditionally demands the Hamiltonian be Hermitian for the description of closed systems, non-Hermitian Hamiltonians have proven useful for the description of open systems involving decay, scattering and transport phenomena [1–6]. Here one uses Hamiltonians whose eigenvalues have negative imaginary parts which leads to a monotonic decrease of the overall probability in time. Although in most cases non-Hermitian Hamiltonians are introduced heuristically, they can be incorporated in a mathematically satisfactory way starting from the system coupled to a continuum of states (see [2, 3, 7] and references cited therein). Furthermore, non-Hermitian $\mathcal{PT}$ symmetric Hamiltonians

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can have a purely real spectrum in certain parameter regions and can be used to define a consistent description of closed quantum systems [8–10]. Non-Hermitian quantum dynamics is sometimes regarded as a mere perturbation to Hermitian dynamics, adding an overall decay. However, non-Hermitian dynamics is in general very different from the familiar unitary time evolution. Much of this difference is due to the non-orthogonality of the non-Hermitian eigenfunctions, which can even degenerate at 'exceptional points' in a parameter space. The various intriguing effects associated with non-Hermitian quantum evolution attracted considerable attention recently, both from theoretical and from experimental side [11–19]. The field is still in an early stage of its development and many generic features are far from being fully understood.

Little is known about the quantum-classical correspondence of systems that are open in the above-mentioned sense. Considerable insights have recently been gained from the study of toy models described by subunitary or truncated unitary operators associated with open quantum maps, where a fractal Weyl law has been found [20–22]. In the general case, however, even the classical analogue of non-Hermitian quantum theories has hitherto remained an open question. Complex extension of classical dynamics in terms of complex coordinates and momenta has been studied in the contexts of both Hermitian and non-Hermitian theories [23–26]. But these systems cannot be regarded as representing the classical limits of non-Hermitian quantum systems for which momentum and coordinate operators have real expectation values. On the other hand, in the attempts to quantizing classical damped motion, a possible link to non-Hermitian Hamiltonians has been observed [27–29], but studies in this direction have not been substantiated, there is no generic structure available and none of the approaches have been able to provide the desired quantum-classical correspondence in a rigorous sense. Recently, a generalized classical approximation using algebraic coherent states [30–32] for a non-Hermitian Bose–Hubbard dimer [33, 34] was introduced in [35] in the context of Bose–Einstein condensates.

Here we show that there is a generalized canonical structure that can be derived as the classical limit of non-Hermitian quantum dynamics. This structure incorporates a metric gradient flow generated by the anti-Hermitian part of the Hamiltonian. It is closely related to the canonical formulation of classical damped motion suggested in the past [36–38], as well as to the gradient flow appearing in thermal and statistical physics and control theory [39, 40]. It should be emphasized that we do not start from a classical damped system and quantize it. We rather begin with the non-Hermitian quantum system and perform a coherent state approximation that is often used to define the classical limit in the Hermitian case. The generalized canonical structure is derived for arbitrary quantum systems which have a Euclidean phase space. The Bose–Hubbard system studied in [35], however, can be regarded as an angular momentum system and thus the classical phase space is given by the Bloch sphere. Here we show that the corresponding classical dynamics can nevertheless be formulated according to the proposed generalized Hamiltonian structure. This gives some evidence that the proposed structure persists for more general phase space geometries.

The paper is organized as follows. In section 2 we give a brief introduction to non-Hermitian quantum dynamics, before we introduce the classical limit using coherent states in section 3. We establish the generalized canonical structure for arbitrary one-dimensional quantum systems on a flat phase space, using Glauber coherent states. This derivation is analogous for higher dimensions. In section 4 we present a case study for a damped oscillator. In section 5, we briefly review the classical approximation [35] for the angular momentum system studied in [33, 34] and show that the generalized canonical structure can be used to describe the classical dynamics on the spherical phase space. We end with a summary and short outlook.

2. Non-Hermitian quantum dynamics

For a general quantum system with Hamiltonian $\hat{\mathcal{H}}$, not necessarily Hermitian, the dynamics of a pure state $|\Psi\rangle$ is governed by the Schrödinger equation

$$\mathrm{i}\hbar|\dot{\Psi}\rangle = \hat{\mathcal{H}}|\Psi\rangle. \tag{1}$$

We can decompose the Hamiltonian into a Hermitian and an anti-Hermitian part via $\hat{\mathcal{H}} = \hat{H} - \mathrm{i}\hat{\Gamma}$, with $\hat{H} = \hat{H}^\dagger$ and $\hat{\Gamma} = \hat{\Gamma}^\dagger$, where the damping term $\hat{\Gamma}$ is assumed to be non-negative. The special choice of the anti-Hermitian part $\hat{\Gamma}$ depends on the dynamics to be modelled, and the physical interpretation as a damping will become clear later. Under time evolution all states decay towards the subspace spanned by the eigenvectors of $\hat{\mathcal{H}}$ with the smallest decay rate (imaginary part of the eigenvalue). Assuming a time-independent Hamiltonian with a discrete spectrum, the time evolution can be expressed as

$$|\psi(t)\rangle = \sum_n c_n\,\mathrm{e}^{-\mathrm{i}E_n t/\hbar}|\varphi_n\rangle. \tag{2}$$

Here $\hat{\mathcal{H}}|\varphi_n\rangle = \mathcal{E}_n|\varphi_n\rangle$, $\mathcal{E}_n = E_n - \mathrm{i}\Gamma_n$ with real $E_n \leqslant E_{n+1}$ and non-negative Γ_n. The expansion coefficients c_n can be obtained from the initial state by projection onto the left eigenstates, i.e. the eigenstates of $\hat{\mathcal{H}}^\dagger$. If there are no real eigenvalues, every initial state approaches the most stable initially populated eigenstate in the long time limit:

$$|\psi(t)\rangle \longrightarrow c_0\,\mathrm{e}^{-\Gamma_0 t/\hbar}\,\mathrm{e}^{-\mathrm{i}E_0 t/\hbar}|\varphi_0\rangle, \tag{3}$$

where we have assumed that this is the (non-degenerate) ground state, and the norm approaches

$$n(t) = \langle\psi(t)|\psi(t)\rangle \longrightarrow |c_0|^2\,\mathrm{e}^{-2\Gamma_0 t/\hbar}\langle\varphi_0|\varphi_0\rangle. \tag{4}$$

From the non-unitary state evolution (1), one can immediately deduce a generalized Heisenberg equation of motion [2] for the diagonal matrix element of an operator $\hat{A}$ (for simplicity we consider only the case $\partial\hat{A}/\partial t = 0$):

$$\begin{aligned}\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi|\hat{A}|\psi\rangle &= \langle\psi|\hat{A}\hat{\mathcal{H}} - \hat{\mathcal{H}}^\dagger\hat{A}|\psi\rangle\\ &= \langle\psi|\,[\hat{A},\hat{H}]|\psi\rangle - \mathrm{i}\langle\psi|\,[\hat{A},\hat{\Gamma}]_+|\psi\rangle,\end{aligned} \tag{5}$$

where $[\;,\;]_+$ is the anti-commutator. Taking into account the fact that the norm is not conserved but decays according to

$$\hbar\frac{\mathrm{d}n}{\mathrm{d}t} = -2\langle\psi|\hat{\Gamma}|\psi\rangle, \tag{6}$$

the dynamical equation of motion for the expectation value of an operator $\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle/\langle\psi|\psi\rangle$ reads

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle = \langle[\hat{A},\hat{H}]\rangle - 2\mathrm{i}\,\Delta^2_{A\Gamma}, \tag{7}$$

with the covariance $\Delta^2_{AB} = \left\langle\frac{1}{2}[\hat{A},\hat{B}]_+\right\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$. In particular, for $\hat{A} = \hat{H}$ we have

$$\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{H}\rangle = -2\,\Delta^2_{H\Gamma}. \tag{8}$$

This evolution equation simplifies in the special case $\hat{\Gamma} = k\hat{H}$ with a non-negative real constant k to

$$\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{H}\rangle = -2(\langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2). \tag{9}$$

This equation has been suggested by Gisin [41, 42], however, with no connection to the non-Hermitian Schrödinger equation (1). Instead, it was derived from the nonlinear quantum evolution equation

$$\mathrm{i}\hbar|\dot{\phi}\rangle = \mathrm{H}|\phi\rangle - \mathrm{i}(\langle\hat{\Gamma}\rangle - \hat{\Gamma})|\phi\rangle = \mathrm{H}|\phi\rangle - \mathrm{i}[|\phi\rangle\langle\phi|, \hat{\Gamma}]|\phi\rangle \tag{10}$$

with $\langle\hat{A}\rangle = \langle\phi|\hat{A}|\phi\rangle$ [41–45]. One can easily see that there is a direct relation between the nonlinear dissipative quantum evolution equation (10) and the non-Hermitian Schrödinger equation (1). A transformation to a renormalized state vector $|\phi\rangle = |\psi\rangle/\sqrt{n}$ using (1) and (6), i.e. $\hbar\dot{n} = -2\langle\hat{\Gamma}\rangle n$, immediately leads to (10). Note that these nonlinear evolution equations can also be expressed in terms of the density operator (see, e.g., [41, 42]), where they take the form of a double bracket flow frequently investigated in control theory and thermal physics [46–48]. Furthermore, they can be extended to mixed states evolving towards thermal equilibrium [49–52]. Here, however, we intend to investigate the classical limit of the non-Hermitian evolution (for a pure state), a topic only briefly addressed in [41].

3. A generalized classical limit for one-dimensional quantum dynamics

Often the classical counterpart of a quantum system is obtained in a rather sloppy manner by simply 'taking the hats off the operators' and identifying the resulting quantities with the classical ones. For a one-dimensional quantum system describing a particle of mass m with momentum $\hat{p}$ and position $\hat{q}$, one can conveniently express the Hamiltonian in terms of harmonic oscillator ladder operators

$$\hat{a} = \frac{m\omega\hat{q} + \mathrm{i}\hat{p}}{\sqrt{2m\hbar\omega}}, \qquad \hat{a}^{\dagger} = \frac{m\omega\hat{q} - \mathrm{i}\hat{p}}{\sqrt{2m\hbar\omega}}. \tag{11}$$

Coherent states $|\alpha\rangle$ are defined by

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \qquad \alpha = \frac{m\omega q + \mathrm{i}p}{\sqrt{2m\hbar\omega}} \tag{12}$$

with $q = \langle\alpha|\hat{q}|\alpha\rangle$ and $p = \langle\alpha|\hat{p}|\alpha\rangle$, and the minimal uncertainty product $\Delta q\Delta q = \hbar/2$, which can be associated with a phase space point in the limit $\hbar \to 0$.

Applying the identification

$$\sqrt{\hbar}\hat{a} \to z = \frac{m\omega q + \mathrm{i}p}{\sqrt{2m\omega}} \qquad \text{and} \qquad \sqrt{\hbar}\hat{a}^{\dagger} \to z^* = \frac{m\omega q - \mathrm{i}p}{\sqrt{2m\omega}}, \tag{13}$$

replacing all operators in the Hamiltonian with their associated c-numbers, and then identifying the result with a Hamiltonian function on the classical phase space, one obtains the classical counterpart in this formulation. The Heisenberg equations of motion for an operator $\hat{F}$ are implicitly replaced with the canonical equation of motion in terms of Poisson brackets for the associated function $F(q, p)$ of the classical canonical variables:

$$\dot{\hat{F}} = \frac{1}{\mathrm{i}\hbar}[\hat{F}, \hat{H}] \quad \to \quad \dot{F} = \frac{1}{\mathrm{i}}\{F, H\}_{z,z^*} = \{F, H\}_{q,p}, \tag{14}$$

with Poisson brackets $\{\,,\,\}$. Here we made use of the complex formulation of classical mechanics in terms of z and z^* [53]. This classical approximation can be motivated in the following way. For an operator function $\hat{F}$ the relations

$$[\hat{a}, \hat{F}] = \frac{\partial\hat{F}}{\partial\hat{a}^{\dagger}} \qquad \text{and} \qquad [\hat{a}^{\dagger}, \hat{F}] = -\frac{\partial\hat{F}}{\partial\hat{a}} \tag{15}$$

hold [54]. Therefore, we can express the Heisenberg equations of motion for the ladder operators in the form

$$\mathrm{i}\hbar\dot{\hat{a}} = \frac{\partial\hat{H}}{\partial\hat{a}^{\dagger}} \qquad \text{and} \qquad \mathrm{i}\hbar\dot{\hat{a}}^{\dagger} = -\frac{\partial\hat{H}}{\partial\hat{a}}. \tag{16}$$

We can then think of the identification (13) as replacing the quantum operators by their expectation values in coherent states, for which expectation values can be factorized. This immediately yields the canonical equations of motion for z and z^*, and the corresponding equations of motion for a function of the canonical variables in terms of the Poisson brackets. Thus, the classical approximation can be rephrased as the assumption that an initially coherent state stays coherent throughout the time evolution. This is analogous to the spirit of the so-called frozen Gaussian approximation [55, 56] for the dynamics of a quantum system. Note that the approximation becomes exact if the Hamiltonian is linear in the group operators $\hat{a}^\dagger\hat{a}$, $\hat{a}$ and $\hat{a}^\dagger$ which can be verified by explicit calculation of the action of the resulting time evolution operator on an initially coherent state. In addition in a semiclassical context one often symmetrizes the appearing ladder operators to account for the zero point energy. Here, however, we deal with the *classical* approximation for $\hbar \to 0$ where the zero point energy vanishes identically and thus, we neglect issues of operator ordering. Furthermore, while an additional constant changes the energy values, it leaves the dynamics invariant.

Let us now derive the general structure of the classical equations of motion for a non-Hermitian Hamiltonian. For this we have to translate the equations of motion for the expectation values of the ladder operators for an arbitrary Hamiltonian to their classical counterparts using the coherent state approximation. We can express the anti-commutators appearing in the equation of motion, making use of the identity [54]

$$[\hat{a}, \hat{F}]_+ = 2\hat{F}\hat{a} + \frac{\partial \hat{F}}{\partial \hat{a}^\dagger}, \qquad [\hat{a}^\dagger, \hat{F}]_+ = 2\hat{a}^\dagger\hat{F} + \frac{\partial \hat{F}}{\partial \hat{a}} \tag{17}$$

for an operator function $\hat{F} = \hat{F}(\hat{a}, \hat{a}^\dagger)$. Inserting expressions (17) in the generalized Heisenberg equations of motion (7), we find

$$\begin{aligned} \mathrm{i}\hbar\frac{d}{dt}\langle\hat{a}\rangle &= \left\langle\frac{\partial \hat{H}}{\partial \hat{a}^\dagger}\right\rangle - \mathrm{i}\left(\left\langle 2\hat{\Gamma}\hat{a} + \frac{\partial\hat{\Gamma}}{\partial\hat{a}^\dagger}\right\rangle - 2\langle\hat{a}\rangle\langle\hat{\Gamma}\rangle\right) \\ \mathrm{i}\hbar\frac{d}{dt}\langle\hat{a}^\dagger\rangle &= -\left\langle\frac{\partial \hat{H}}{\partial \hat{a}}\right\rangle - \mathrm{i}\left(\left\langle 2\hat{a}^\dagger\hat{\Gamma} + \frac{\partial\hat{\Gamma}}{\partial\hat{a}}\right\rangle - 2\langle\hat{a}^\dagger\rangle\langle\hat{\Gamma}\rangle\right). \end{aligned} \tag{18}$$

Applying the coherent state approximation hence yields the desired classical equations of motion governed by the complex valued Hamiltonian function $\mathcal{H} = H - \mathrm{i}\Gamma$, where H and Γ are the Weyl symbols of the operators (i.e. the expectation values in coherent states):

$$\mathrm{i}\dot{z} = \frac{\partial H}{\partial z^*} - \mathrm{i}\frac{\partial\Gamma}{\partial z^*} = \frac{\partial\mathcal{H}}{\partial z^*}, \qquad \mathrm{i}\dot{z}^* = -\frac{\partial H}{\partial z} - \mathrm{i}\frac{\partial\Gamma}{\partial z} = -\frac{\partial\mathcal{H}^*}{\partial z}. \tag{19}$$

The evolution equation (6) for the norm is then approximated by

$$\hbar\dot{n} = -2(\Gamma + \Gamma_0)n, \tag{20}$$

with constant Γ_0 (note that $\Gamma = \Gamma(z, z^*)$ depends implicitly on time). This evolution equation for the norm, in fact, goes beyond a purely classical description because it depends on $\hbar$. Furthermore, as will be clear from the examples below, Γ tends to zero in the long time limit and the constant Γ_0 accounts for the finite asymptotic decay rate (4).

Translated to q and p the equations of motion (19) can be expressed as

$$\dot{q} = \frac{\partial H}{\partial p} - \frac{1}{m\omega}\frac{\partial\Gamma}{\partial q} \qquad \text{and} \qquad \dot{p} = -\frac{\partial H}{\partial q} - m\omega\frac{\partial\Gamma}{\partial p}. \tag{21}$$

For the time evolution of a dynamical variable $A(q, p)$ with $\partial A/\partial t = 0$ we therefore find

$$\begin{aligned}\dot{A}(q,p) &= \frac{\partial A}{\partial q}\dot{q} + \frac{\partial A}{\partial p}\dot{p}\\ &= \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q} - \frac{1}{m\omega}\frac{\partial A}{\partial q}\frac{\partial \Gamma}{\partial q} - m\omega\frac{\partial A}{\partial p}\frac{\partial \Gamma}{\partial p}\\ &= \{A,\ H\}_{q,p} - \left(\frac{1}{m\omega}\frac{\partial A}{\partial q}\frac{\partial \Gamma}{\partial q} + m\omega\frac{\partial A}{\partial p}\frac{\partial \Gamma}{\partial p}\right).\end{aligned} \tag{22}$$

Equation (21) can be represented in a generalized canonical structure. To see this we introduce phase space variables of equal dimension via the canonical transformation

$$q \to q/\sqrt{m\omega}, \qquad p \to \sqrt{m\omega}\, p. \tag{23}$$

Then equation (21) can be written in the form

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \Omega^{-1}\nabla H - G^{-1}\nabla\Gamma, \tag{24}$$

where ∇ is the phase space gradient operator, Ω is the symplectic unit matrix and G denotes the phase space metric. Since the considered phase space has Euclidean geometry, we thus have

$$\Omega = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad G = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{25}$$

(see appendix A and [57] for further details).

The dynamical equation (24) is a combination of a canonical symplectic flow generated by the real part H of the Hamiltonian function and a canonical metric gradient flow generated by the imaginary part Γ of the Hamiltonian function. The symplectic part evidently gives rise to the familiar Hamiltonian dynamics of classical mechanics. The appearance of the gradient flow may at first seem surprising in the present context. However, if we recall the fact that the gradient vector with a negative sign points in the direction of the steepest descent of the function Γ, we see that this part of the dynamics drives the system towards the minimum of Γ and thus can naturally be associated with a damping (see, e.g., [36–40]). We shall show in section 5 that this structure persists for a spherical phase space arising as the classical limit of the quantum angular momentum system studied in [35, 57].

Finally, it should be pointed out that the frequency ω appearing in the classical equations of motion, i.e. the frequency of the harmonic oscillator introduced to define the coherent states, is still a parameter and can be adjusted, for example, to improve the quality of the classical approximation.

In summary, we identified a generalized classical canonical structure (24) associated with non-Hermitian dynamics for a flat phase space. We derived this structure from a coherent state approximation and it should be noted that it does not depend on the specific choice of $\hat{H}$ and $\hat{\Gamma}$. In the case where both are linear in the generators of the Heisenberg–Weyl algebra, $\hat{a}$, $\hat{a}^\dagger$ and $\hat{a}\hat{a}^\dagger$, and for a coherent initial state, this approximation becomes exact and the quantum dynamics is fully captured by the classical equations of motion. To obtain further insight into the quantum classical correspondence for non-Hermitian systems, we shall present some example studies in the following section.

4. The damped oscillator

In this section we will study the dynamics of a damped oscillator starting with a purely harmonic case. Here the classical approximation is exact, provided that the state is initially coherent. This is a well-known correspondence identity for Hermitian systems which also extends to the non-Hermitian case. In the subsequent section we present some results of a case study of an anharmonic oscillator, where the classical dynamics is approximate.

4.1. The damped harmonic oscillator

Let us first consider the purely harmonic case

$$\hat{H}_0 = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2\,, \tag{26}$$

and assume that the non-Hermitian part of the Hamiltonian is

$$\hat{\Gamma} = k\hat{H}_0, \tag{27}$$

with a positive real number k. Thus, the metric gradient part of the resulting dynamics (24) will drive the system towards the minimum of the energy. The classical equations of motion (21) are

$$\dot{q} = p/m - \gamma q, \qquad \dot{p} = -m\omega^2 q - \gamma p, \tag{28}$$

with $\gamma = k\omega$. Eliminating the momentum p, we obtain

$$\ddot{q} + 2\gamma\dot{q} + (\omega^2 + \gamma^2)q = 0, \tag{29}$$

which is the familiar classical equation of motion for a damped harmonic oscillator, however, with a frequency $\omega_0 = \sqrt{\omega^2 + \gamma^2}$, which implies that the damping is always subcritical. The evolution equation for the damped harmonic oscillator (29) has been presented by Gisin in [41].

Here the Hamiltonian $\hat{\mathcal{H}}$ is linear in the generators $\hat{a}$, $\hat{a}^\dagger$ and $\hat{a}^\dagger\hat{a} + 1/2$ of the harmonic oscillator algebra. Hence, an initially coherent state remains coherent and the classical evolution equation for q and p agrees *exactly* with the quantum evolution of the expectation values $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$.

Note, however, that this is no longer the case if one uses (i) coherent states in the classicalization belonging to a harmonic oscillator with frequency $\omega' \neq \omega$; or (ii) a different damping operator $\hat{\Gamma}$, for example, $\hat{\Gamma} = k\hat{p}^2/2m$ (physically that means that the anti-Hermitian part seeks to minimize the kinetic energy only). In the first case one obtains the classical evolution equation

$$\ddot{q} + \gamma\left(\frac{\omega'}{\omega} + \frac{\omega'}{\omega}\right)\dot{q} + (\omega^2 + \gamma^2)q = 0, \tag{30}$$

and in the second case one finds

$$\ddot{q} + 2\gamma\dot{q} + \omega^2 q = 0. \tag{31}$$

Both results are now approximations of the true quantum dynamics because the Hamiltonian is no longer linear in the generators of the algebra. The same classical equation of motion also appears in the dynamics of a Caldirola–Kanai Hamiltonian which describes the classical dynamics of a harmonic oscillator whose mass increases exponentially in time. This Hamiltonian can immediately be quantized (see the recent review [58]) and yields dynamics different from the one investigated here.

In what follows, we choose the proper coherent states and $\hat{\Gamma}$ as given in (27). In this case, the Hamiltonian can be simply written as

$$\hat{\mathcal{H}} = \hbar(\omega - \mathrm{i}\gamma)(\hat{a}^\dagger\hat{a} + 1/2), \tag{32}$$

a harmonic oscillator with a complex frequency $\widetilde{\omega} = \omega - \mathrm{i}\gamma$, and many results derived for the Hermitian harmonic oscillator are also valid here. For instance, the time evolution of an initially coherent state $|\alpha_0\rangle$ is given by (see, e.g., [54])

$$|\psi(t)\rangle = \mathrm{e}^{\mathrm{i}\widetilde{\omega}t/2 + D_t\alpha_0\hat{a}^\dagger}|\alpha_0\rangle = \mathrm{e}^{\mathrm{i}\widetilde{\omega}t/2 - |\alpha_0|^2(1-\mathrm{e}^{-2\gamma t})/2}|\alpha_t\rangle \tag{33}$$

with

$$D_t = \mathrm{e}^{-\mathrm{i}\widetilde{\omega}t} - 1, \tag{34}$$

where $\alpha_t = (1 + D_t)\alpha_0 = \mathrm{e}^{-\mathrm{i}\widetilde{\omega}t}\alpha_0$ satisfies the (classical!) equation of motion $\mathrm{i}\dot{\alpha}_t = \widetilde{\omega}\alpha$. The time dependence of the norm goes exponentially to zero for long times,

$$n_t = \langle\psi(t)|\psi(t)\rangle = \mathrm{e}^{-\gamma t - |\alpha_0|^2(1-\mathrm{e}^{-2\gamma t})} \to \mathrm{e}^{-\gamma t} \to 0, \tag{35}$$

whereas for short times it decays linearly according to $n_t \approx 1 - \gamma(1 + |\alpha_0|^2)t$. Note that the time derivative of n_t agrees exactly with the classical equation (20) for $\Gamma_0 = k\hbar\gamma/2$, which justifies the ad hoc insertion of this correction term.

4.2. *The forced and damped harmonic oscillator*

We now consider one of the most celebrated dynamical systems in textbooks, the (classical) damped harmonic oscillator with harmonic driving

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = F_0 \cos\Omega t. \tag{36}$$

Here all trajectories approach the limit cycle

$$q(t) = Q\cos(\Omega t - \delta), \tag{37}$$

where the amplitude and phase shift are

$$Q^2 = \frac{F_0^2}{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}, \qquad \delta = \arctan\frac{2\gamma\Omega}{\omega_0^2 - \Omega^2}. \tag{38}$$

The corresponding quantum system has been studied only rarely (see, e.g., [44, 50]). Here we consider the Hamiltonian

$$\mathcal{H} = \hbar\widetilde{\omega}(\hat{a}^\dagger\hat{a} + 1/2) + \hbar f_t\hat{a} + \hbar f_t^*\hat{a}^\dagger, \tag{39}$$

again with a complex frequency $\widetilde{\omega}$ and a time dependent driving f_t. The closed form solution known for real frequency (see, e.g., [54, section 3.11]) can be extended to the non-Hermitian case with complex frequency $\widetilde{\omega}$. For convenience we provide the solution here for an initially coherent state $|\alpha_0\rangle$:

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\widetilde{\omega}t/2 + A_t + B_t\alpha_0 + (C_t + D_t\alpha_0)\hat{a}^\dagger}|\alpha_0\rangle = \mathrm{e}^{-\mathrm{i}\widetilde{\omega}t/2 + A_t + B_t\alpha_0 - |\alpha_0|^2/2 + |\alpha_t|^2/2}|\alpha_t\rangle, \tag{40}$$

with $\alpha_t = C_t + (1 + D_t)\alpha_0$. The parameters satisfy the differential equations

$$\mathrm{i}\dot{D} = \widetilde{\omega}(D + 1), \quad \mathrm{i}\dot{B} = f_t(D + 1), \quad \mathrm{i}\dot{C} = \widetilde{\omega}C + f_t^*, \quad \mathrm{i}\dot{A} = f_t C, \tag{41}$$

with initial conditions $A_0 = B_0 = C_0 = D_0 = 0$. The solutions are

$$B_t = -\mathrm{i}\int_0^t \mathrm{d}t'\mathrm{e}^{-\mathrm{i}\widetilde{\omega}t'} f_{t'}\,, \quad C_t = -\mathrm{i}\int_0^t \mathrm{d}t'\,\mathrm{e}^{\mathrm{i}\widetilde{\omega}(t'-t)} f_{t'}^*\,, \quad A_t = -\mathrm{i}\int_0^t \mathrm{d}t'\, f_{t'}C_{t'} \tag{42}$$

and D_t is again given by (34). Note that α_t, parametrizing the coherent state, is a solution of the differential equations

$$\mathrm{i}\dot{\alpha} = \widetilde{\omega}\alpha + f_t^*, \qquad \mathrm{i}\dot{\alpha}^* = -\widetilde{\omega}^*\alpha^* - f_t, \tag{43}$$

which are indeed the classical equation of motion

$$\mathrm{i}\dot{\alpha} = \frac{\partial\mathcal{H}}{\partial\alpha^*}, \qquad \mathrm{i}\dot{\alpha}^* = -\frac{\partial\mathcal{H}^*}{\partial\alpha} \tag{44}$$

for the Hamiltonian $\mathcal{H} = \widetilde{\omega}\alpha^*\alpha + f_t\alpha + f_t^*\alpha^*$, where α and α^* act as canonical coordinate and momentum, respectively. For the special case of harmonic driving, $f_t = f_0\cos\Omega t$, one obtains

$$B_t = \frac{f_0}{\widetilde{\omega}^2 - \Omega^2}\{(\widetilde{\omega} + \Omega)\,\mathrm{e}^{-\mathrm{i}(\widetilde{\omega}-\Omega)t} + (\widetilde{\omega} - \Omega)\,\mathrm{e}^{-\mathrm{i}(\widetilde{\omega}+\Omega)t} - 2\widetilde{\omega}\}, \tag{45}$$

$$C_t = -\frac{f_0}{2(\widetilde{\omega}^2 - \Omega^2)}\{(\widetilde{\omega} - \Omega)\,\mathrm{e}^{\mathrm{i}\Omega t} + (\widetilde{\omega} + \Omega)\,\mathrm{e}^{-\mathrm{i}\Omega t} - 2\widetilde{\omega}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}t}\}, \tag{46}$$

$$\begin{aligned} A_t = \frac{f_0^2}{4(\widetilde{\omega}^2 - \Omega^2)}\Bigg\{&2\mathrm{i}\widetilde{\omega}t + \frac{1}{2\Omega}((\widetilde{\omega} - \Omega)\,\mathrm{e}^{2\mathrm{i}\Omega t} - (\widetilde{\omega} + \Omega)\,\mathrm{e}^{-2\mathrm{i}\Omega t} + 2\Omega) \\ &+ \frac{2\widetilde{\omega}}{\widetilde{\omega}^2 - \Omega^2}((\widetilde{\omega} + \Omega)\,\mathrm{e}^{-\mathrm{i}(\widetilde{\omega}-\Omega)t} + (\widetilde{\omega} - \Omega)\,\mathrm{e}^{-\mathrm{i}(\widetilde{\omega}+\Omega)t} - 2\widetilde{\omega})\Bigg\}, \end{aligned} \tag{47}$$

which approach, in the long time limit, to the simpler expressions $D_t \to -1$, $B_t \to 0$, $A_t \to A_t^{\infty} + A_t^{(0)}$, with

$$A_t^{\infty} = \frac{\mathrm{i}f_0^2}{2(\widetilde{\omega}^2 - \Omega^2)}\widetilde{\omega}t, \tag{48}$$

$$A_t^{(0)} = \frac{f_0^2}{4(\widetilde{\omega}^2 - \Omega^2)}\left\{\frac{1}{2\Omega}((\widetilde{\omega} - \Omega)\,\mathrm{e}^{2\mathrm{i}\Omega t} - (\widetilde{\omega} + \Omega)\,\mathrm{e}^{-2\mathrm{i}\Omega t})\right\}, \tag{49}$$

and

$$C_t \to -\frac{f_0}{2(\widetilde{\omega}^2 - \Omega^2)}\{(\widetilde{\omega} - \Omega)\,\mathrm{e}^{\mathrm{i}\Omega t} + (\widetilde{\omega} + \Omega)\,\mathrm{e}^{-\mathrm{i}\Omega t}\} = \alpha_t^{(0)} \tag{50}$$

with $\alpha_t \to \alpha_t^{(0)}$, the limit cycle (37). We note that $\alpha_t^{(0)}$ and $A_t^{(0)}$ are T-periodic and the limiting state for $t \to \infty$ can be written in the form

$$|\psi_0(t)\rangle = \mathrm{e}^{-\mathrm{i}\epsilon_0 t/\hbar}|\phi_0(t)\rangle \qquad \text{with} \qquad |\phi_0(t)\rangle = c\,\mathrm{e}^{-\mathrm{i}A_t^{(0)} + |\alpha_t|^2/2}\big|\alpha_t^{(0)}\big\rangle = |\phi_0(t+T)\rangle, \tag{51}$$

where c is a constant, and

$$\epsilon_0 = \frac{\hbar\widetilde{\omega}}{2}\left(1 - \frac{f_0^2}{\widetilde{\omega}^2 - \Omega^2}\right). \tag{52}$$

The quantum dynamics can be computed numerically, e.g. in the discrete operator representation [59]. As an example, figure 1 shows the evolution of a coherent state initially located at $(q_0, p_0) = (2, 0)$. The parameters are chosen as $\omega = 1$, $m = 1$ (these are kept fixed in all calculations) and $\gamma = 0.1$, $\hbar = 1$. For the driving force we used, $\Omega = 1$ and $f_0 = 0.1$. The time dependence of the expectation value $\langle\hat{q}\rangle$ shown as a solid black curve agrees, of course, with the analytic formula given above and also with the classical oscillation $q(t)$. Asymptotically, the motion approaches the limit cycle (37).

If the initial state is chosen differently from a coherent one, the agreement between classical and quantum evolution breaks down, as in the Hermitian case. As an example, the dashed blue curve in the figure shows the results for an initial superposition of two coherent states located at $(q_0, p_0) = (2, 0)$ and $(-2, 0)$ (a so-called cat state). Here $\langle\hat{q}\rangle$, which is initially equal to zero, cannot be described by a single classical trajectory but shows an interference structure before it converges to the asymptotic limit state. This behaviour is characteristically different from the Hermitian case, where no limit cycle exists and the quantum evolution never approaches the classical dynamics.

Let us finally remark that in the quantum case most states approach asymptotically to a unique state, a quantum limit cycle. This is evident, if one formulates the dynamics in terms

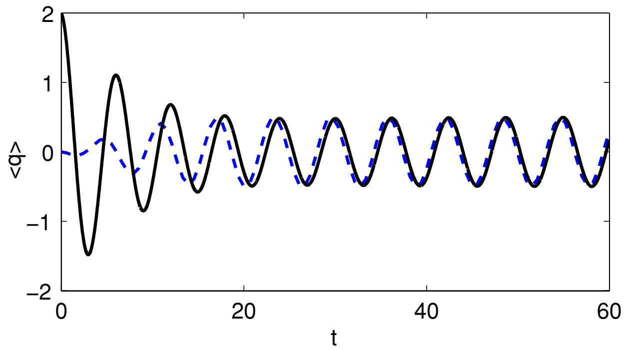

Figure 1. Time evolution of the mean position $\langle \hat{q} \rangle$ for a damped quantum harmonic oscillator ($\omega = m = \hbar = 1$, $\gamma = 0.1$) which is harmonically driven ($f_0 = 0.1$, $\Omega = 1$). The solid black curve shows the evolution of a coherent state initially located at $(q_0, p_0) = (2, 0)$. This curve agrees with the classical motion and converges to the limit cycle. The dashed blue curve is obtained for an initial superposition of two coherent states located at $(2, 0)$ and $(-2, 0)$.

of non-Hermitian Floquet states. Here the time evolution can be expressed in the form of a linear combination as in (2), where the eigenstates are replaced by the Floquet states

$$|\psi_n(t)\rangle = \mathrm{e}^{-\mathrm{i}\epsilon_n t/\hbar}|\varphi_n(t)\rangle \qquad \text{with} \qquad |\varphi_n(t+T)\rangle = |\varphi_n(t)\rangle \tag{53}$$

and the eigenvalues by the (complex) quasienergies ϵ_n. Then each state approaches asymptotically the most stable Floquet state, which can be identified as the quantum limit cycle. In the present case of a forced harmonic oscillator, the quasienergies are known analytically (see, e.g., [59, 60]) and the solution can be directly extended to complex frequencies:

$$\epsilon_n = \hbar\widetilde{\omega}\left(n + \frac{1}{2} - \frac{f_0^2}{2(\widetilde{\omega}^2 - \Omega^2)}\right), \qquad n = 0, 1, \ldots, \tag{54}$$

which agrees for $n = 0$ with the result already given in (52). The most stable state $n = 0$ is, of course, only reached if the projection of the initial state onto this state is different from zero. Otherwise, the state approaches the most stable initially populated quasienergy state. In fact, all quasienergy states appear as quantum limit cycles, as already remarked by Gisin [44].

As an illustration, figure 2 shows an example of the quantum limit cycle in phase space. Shown in false colours is the Husimi distribution, i.e. the projection on coherent states

$$P(p, q; t) = |\langle\alpha|\psi(t)\rangle|^2, \qquad \alpha = (m\omega q + \mathrm{i}p)/\sqrt{2m\hbar\omega} \tag{55}$$

which is averaged over one period of the driving in the long time limit. Also shown is a classical trajectory started at $p_0 = 0$, $q_0 = 2$ which converges to the classical limit cycle that coincides with the ridge of the quantum distribution.

4.3. *A damped anharmonic oscillator*

As an example of an anharmonic oscillator we consider the Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2 + \frac{\beta}{4}\hat{q}^4, \tag{56}$$

where the non-Hermitian part $\hat{\Gamma}$ of the Hamiltonian is chosen as

$$\hat{\Gamma} = \frac{\gamma}{\omega}\left(\frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2\right) \tag{57}$$

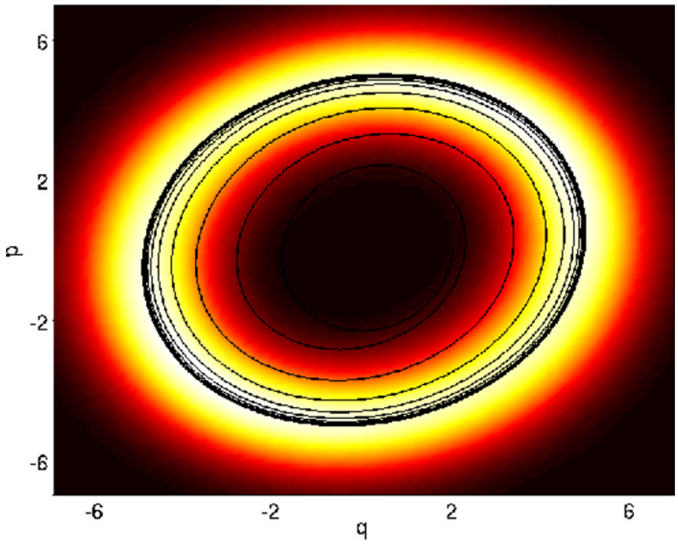

Figure 2. Husimi distribution of the quantum limit cycle for a damped quantum harmonic oscillator ($\omega = m = \hbar = 1$, $\gamma = 0.1$) which is harmonically driven ($f_0 = 1$, $\Omega = 1$), averaged over one driving period. The solid black line shows a classical trajectory started at $(q_0, p_0) = (2, 0)$ for comparison. This curve approaches the classical limit cycle that coincides with the ridge of the averaged Husimi distribution of the quantum limit cycle.

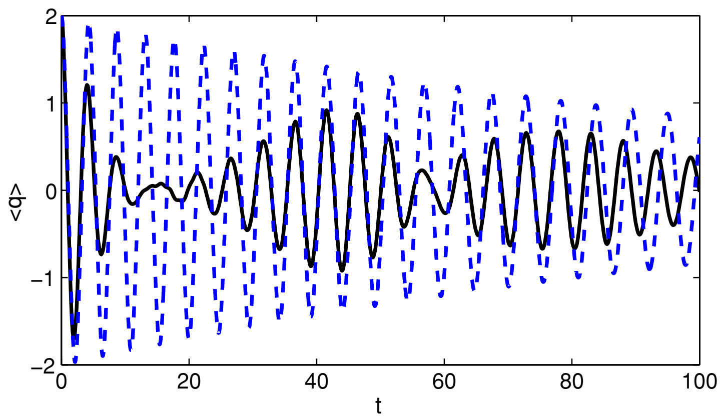

Figure 3. Comparison of the quantum and classical damped anharmonic oscillator with $\beta = 0.4$ and weak damping, $\gamma = 0.01$. The quantum expectation value $\langle q \rangle$ (solid black curve) is shown as a function of time in comparison with the classical evolution $q(t)$ (dashed blue curve).

as for the harmonic oscillator. Then the classical equations of motion (21) are

$$\dot{q} = p/m - \gamma q, \qquad \dot{p} = -m\omega^2 q - \beta q^3 - \gamma p. \tag{58}$$

In the Hermitian case $\gamma = 0$, one finds the well-known deviations between the quantum and classical evolution showing breakdown and revival phenomena. As illustrated in figure 3, these phenomena survive in the case of a weak damping, $\gamma = 0.01$, and moderate anharmonicity

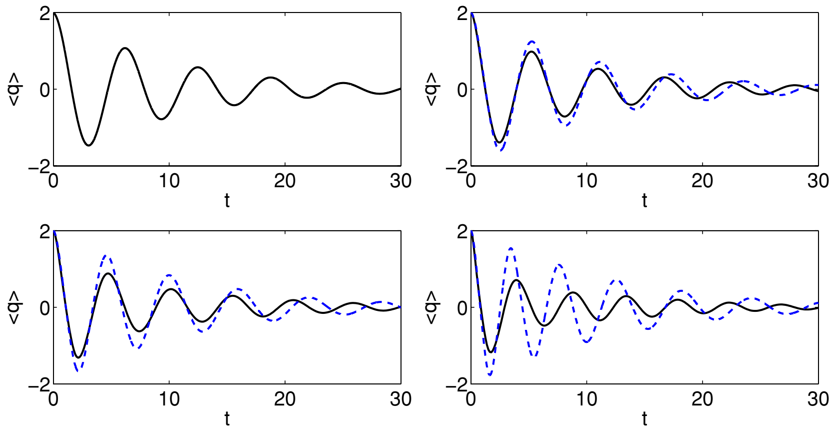

Figure 4. Damped harmonic and anharmonic oscillator: Expectation value $\langle q \rangle$ as a function of time for $\gamma = 0.1$, and $\beta = 0$ (upper left), $\beta = 0.2$ (upper right), $\beta = 0.4$ (lower left) and $\beta = 1.0$ (lower right). The quantum expectation values $\langle q \rangle(t)$ (black solid curves) for $\hbar = 1$ are compared with the classical evolution $q(t)$ (blue dashed curves).

$\beta = 0.4$. The oscillation frequency and the overall decay is reproduced in the classical approximation.

The numerical results for the damped case with stronger damping are shown in figure 4, where we have used the same method and parameters as described above. The damped anharmonic evolution shows differences between the classical and quantum evolution and, as expected, the deviations increase with anharmonicity β. This is due to the increasing deviation of the quantum state from a coherent one during the time evolution. For short times, when the state is still almost coherent, we find a very good agreement between classical and quantum evolution. In further numerical studies, we observed that the qualitative behaviour is not changed if we also chose the anti-Hermitian part $\hat{\Gamma}$ anharmonic. A detailed investigation of the behaviour of the quantum-classical correspondence in dependence on the special choice of Hermitian and anti-Hermitian part as well as their relation is a promising starting point for future studies.

5. An angular momentum system

Recently, a generalized classical approximation in the spirit introduced in the preceding sections has been performed for a non-Hermitian angular momentum system in the context of many-particle mean-field correspondence for ultracold atoms in an open double well trap [35, 57]. The Hamiltonian investigated there is of the form

$$\hat{\mathcal{H}} = 2\epsilon \hat{L}_z + 2v\hat{L}_x + 2c\hat{L}_z^2 - 2\mathrm{i}\gamma(\hat{L}_z + L), \tag{59}$$

where the $\hat{L}_j$ are angular momentum operators and L denotes the angular momentum quantum number. By Schwinger's harmonic oscillator representation of angular momentum, this can be interpreted as a second quantized many-particle model and the classical approximation arises as a mean-field approximation. Here we briefly review the generalized coherent state approximation for this angular momentum system, and show that the resulting mean-field dynamics can be described by the generalized canonical structure (24), however, on a spherical phase space.

To perform the classical approximation we have to replace expectation values in the quantum Heisenberg equations of motion with their values in a coherent state. For angular momentum operators, the classical approximation will be performed using $SU(2)$ coherent states. The classical limit is then realized for large quantum numbers, that is, $L \to \infty$, where we set $\hbar = 1$ for convenience.

The dynamics of the angular momentum expectation values follows from the generalized Heisenberg equation of motion (7):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{L}_x\rangle &= -2\epsilon\langle \hat{L}_y\rangle - 2c\langle[\hat{L}_y, \hat{L}_z]_+\rangle - 4\gamma\, \Delta^2_{\hat{L}_x\hat{L}_z}\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{L}_y\rangle &= 2\epsilon\langle \hat{L}_x\rangle + 2c\langle[\hat{L}_x, \hat{L}_z]_+\rangle - 2v\langle \hat{L}_z\rangle - 4\gamma\, \Delta^2_{\hat{L}_y\hat{L}_z}\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{L}_z\rangle &= 2v\langle \hat{L}_y\rangle - 4\gamma\, \Delta^2_{\hat{L}_z\hat{L}_z},\end{aligned} \tag{60}$$

and the norm of wavefunction decays according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi|\Psi\rangle = -4\gamma\{\langle \hat{L}_z\rangle + L\}\langle\Psi|\Psi\rangle. \tag{61}$$

The generalized $SU(2)$ coherent states, often also denoted as atomic coherent states, can be constructed by an arbitrary $SU(2)$ rotation of the extremal angular momentum state $|L\rangle$ (spin up):

$$|\theta, \phi\rangle = \hat{R}(\theta, \phi)|L\rangle = \mathrm{e}^{\mathrm{i}\theta(\hat{L}_x \sin\phi - \hat{L}_y \cos\phi)}|L\rangle. \tag{62}$$

The coherent state approximation—that is, the assumption that an initially coherent state stays coherent throughout the time evolution—is exact if the Hamiltonian is linear in the generators of the $SU(2)$ algebra, i.e. for $c = 0$ in our case. This can be seen by an explicit calculation of the action of the time evolution operator

$$U(t) = \exp(-\mathrm{i}\hat{H}t/\hbar) \tag{63}$$

on an initially coherent state (62). The classical equations of motion are obtained from the quantum dynamics of the relevant expectation values by replacing all expectation values with their values in coherent states and identify these as the classical quantities. The $SU(2)$ expectation values of the $\hat{L}_j$, $j = x, y, z$ appear as the components of the classical Bloch vector:

$$s_j = \langle \hat{L}_j\rangle/2L. \tag{64}$$

The expectation values of the anti-commutators appearing in (60) in $SU(2)$ states can be shown [35, 57] to factorize as

$$\langle[\hat{L}_i, \hat{L}_j]_+\rangle = 2\left(1 - \frac{1}{2L}\right)\langle \hat{L}_i\rangle\langle \hat{L}_j\rangle + \delta_{ij}L. \tag{65}$$

Inserting these expressions into (60) and taking the classical limit $L \to \infty$ (where we have to keep $2Lc = g$ fixed), we obtain the desired non-Hermitian classical evolution equations:

$$\begin{aligned}\dot{s}_x &= -2\epsilon s_y - 4g s_z s_y + 4\gamma\, s_z s_x,\\ \dot{s}_y &= +2\epsilon s_x + 4g s_z s_x - 2v s_z + 4\gamma\, s_z s_y,\\ \dot{s}_z &= +2v s_y - \gamma\left(1 - 4s_z^2\right).\end{aligned} \tag{66}$$

In the case $g = 0$, in which the assumption that the many-particle state stays coherent in time is exactly fulfilled, these equations exactly coincide with the quantum dynamics for arbitrary L for an initially coherent state. The nonlinear non-Hermitian Bloch equations are real valued,

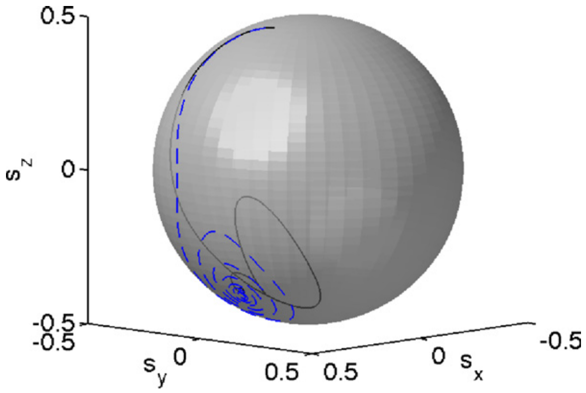

Figure 5. Quantum (black solid line) and classical (blue dashed line) angular momentum evolution on the Bloch sphere. Plotted are the expectation values of the quantum angular momentum components in comparison with the classical Bloch vector for an initially coherent state centred at the classical initial condition close to the north pole of the Bloch sphere up to a time $t_end = 25$ for $\gamma = 0.1$ and $g = 1.5$ and a total angular momentum of $L = 40$.

and conserve $s^2 = s_x^2 + s_y^2 + s_z^2 = 1/4$, i.e. the dynamics is regular and confined to the Bloch sphere. The decay of the total probability can be approximated by

$$\dot{n} = -4\gamma L(2s_z + 1)n. \tag{67}$$

The classical dynamics is analysed in detail in [35, 57]. Here we show that it can be described by the generalized canonical structure (24). For this purpose we introduce the canonical coordinates p and q on the sphere related to the Bloch coordinates via

$$\begin{aligned} s_x &= \tfrac{1}{2}\sqrt{1-p^2}\cos(2q) \\ s_y &= \tfrac{1}{2}\sqrt{1-p^2}\sin(2q) \\ s_z &= \tfrac{1}{2}p. \end{aligned} \tag{68}$$

The equations of motion then read

$$\dot{q} = \epsilon + gp - v\frac{p}{\sqrt{1-p^2}}\cos(2q) \tag{69}$$

$$\dot{p} = -2\gamma(1-p^2) + 2v\sqrt{1-p^2}\sin(2q). \tag{70}$$

This can be expressed as generalized Hamiltonian equations of motion of the form (24) where Ω is the usual symplectic matrix (25), and

$$G = \begin{pmatrix} 2(1-p^2) & 0 \\ 0 & \frac{1}{2(1-p^2)} \end{pmatrix} \tag{71}$$

is the corresponding Kähler metric on the Bloch sphere (see appendix A for some details). The Hamiltonian function $\mathcal{H} = H - \mathrm{i}\Gamma$ is given by the $SU(2)$ expectation value of the quantum Hamiltonian (59):

$$H = \epsilon p + v\sqrt{1-p^2}\cos(2q) + \frac{g}{2}p^2 \qquad \text{and} \qquad \Gamma = \gamma p. \tag{72}$$

In figure 5, we show an example of the quantum dynamics in comparison with the classical evolution on the Bloch sphere for a strong nonlinearity $g = 1.5$ and a small non-Hermiticity $\gamma = 0.1$. Here the classical dynamics has several fixed points, one of them being a sink of the dynamics close to the south pole of the sphere and another being a source close to the north pole. The initial condition in the present example is close to the source. For shorter times the behaviour of the quantum and the classical dynamics is similar. For longer times, however, while the classical dynamics moves towards the sink staying confined on the surface of the Bloch sphere, the quantum dynamics can tunnel through the sphere. Finally they both approach the sink of the dynamics. Note that the quantum classical correspondence for this system can be quite intricate. Depending on the initial conditions and the total angular momentum the quantum dynamics can differ considerably from the underlying classical behaviour due to interference and tunnelling effects. This is analysed in more detail in [57].

6. Conclusions and outlook

In conclusion, we derived a generalized canonical structure (24) incorporating a metric gradient flow that can describe both conservative and dissipative motion as a classical limit of non-Hermitian quantum dynamics. The classical limit was defined as a coherent state approximation and it was shown that the canonical structure arises for the Euclidean phase space geometry associated with Glauber coherent states, irrespective of the special choice of the Hamiltonian. Furthermore, we presented an example of an angular momentum system where the classical limit is given by the limit of large angular momentum. The appropriate coherent states in this context are the $SU(2)$ or atomic coherent states which lead to a spherical classical phase space. We showed that the resulting classical equations of motion can also be described using the generalized canonical structure derived for the Euclidean case. This strongly suggests that the proposed structure holds for more general systems with different metric structures on the classical phase space.

The quantum classical correspondence arising from these non-Hermitian systems is in many cases equivalent to the well-known behaviour observed in Hermitian systems. However, new effects due to the gradient part of the motion are to be expected, the investigation of which is a promising task for future studies. Interesting questions here regard the timescales of the quantum classical correspondence (that is the non-Hermitian equivalents of the Ehrenfest and Heisenberg times for Hermitian systems) and their dependence on the Hermitian and anti-Hermitian part individually as well as the relation of Hermitian and anti-Hermitian part. It is also interesting how the generated classical flow depends on this relation between the Hermitian and anti-Hermitian part of the Hamiltonian and, in particular, which simplifications are to be expected if they are proportional.

Furthermore, the discovery of a general structure in the classical limit of non-Hermitian quantum theories paves the way for the development of new semiclassical techniques such as quantization conditions and trace formulae for these systems.

Acknowledgments

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Appendix A. The metric, the symplectic structure and a generalized canonical evolution on phase spaces

In this appendix, we provide the minimal basic facts concerning the notion of a Kähler structure on manifolds that is relevant for the generalized canonical structure of the non-Hermitian dynamics. The interested reader is referred to [61, 62] for further detail.

In Riemannian geometry, a manifold is equipped with a measure of distance determined by the metric tensor g in the following way. The infinitesimal distance $\mathrm{d}s$ between a point $(x^1, \ldots, x^n)$ and $(x^1 + \mathrm{d}x^1, \ldots, x^n + \mathrm{d}x^n)$ on an n-dimensional manifold X_n is given by the expression

$$\mathrm{d}s^2 = g_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b. \tag{A.1}$$

Here we make use of the Einstein sum convention. The metric tensor thus consists of $n(n+1)/2$ quantities g_{ab} that define the notion of distance on X_n.

Clearly, distance is invariant under local coordinate transformation of the form $(x^1, \ldots, x^n) \to (u^1, \ldots, u^n)$. In other words, if we write $g_{ab}^{(u)}$ for the metric tensor in the transformed $(u^1, \ldots, u^n)$ coordinates, we have

$$\mathrm{d}s^2 = g_{ab}^{(u)}\, \mathrm{d}u^a\, \mathrm{d}u^b = g_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b, \tag{A.2}$$

from which it follows that the metric transforms according to

$$g_{ab}^{(u)} = \frac{\partial x^c}{\partial u^a}\frac{\partial x^d}{\partial u^b}\, g_{cd}. \tag{A.3}$$

If the manifold in question is a classical phase space, then it comes equipped with an additional symplectic structure ω. A space accommodating both a metric g and a symplectic structure ω is said to possess a Kähler structure if these two quantities satisfy the compatibility condition [62], which in the matrix form is written as

$$\omega^{-1} = (g^{-1}\omega g^{-1})^{\mathrm{T}}, \tag{A.4}$$

where the superscript T denotes matrix transpose. It is evident from (A.4) that the compatibility condition is invariant under the simultaneous scale transformation $g \to \kappa g$ and $\omega \to \kappa\omega$ for some $\kappa \neq 0$. Hence, the Kähler metric and symplectic structure are defined only up to a scale factor. On the other hand, if either ω or g is given, then this automatically fixes the relevant scale factor according to the compatibility condition.

In classical mechanics one usually associates the symplectic structure on the phase space with the canonical equations of motion by demanding them to read

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} +\partial H/\partial p \\ -\partial H/\partial q \end{pmatrix} = \Omega^{-1}\vec{\nabla} H, \tag{A.5}$$

and hence fixing the symplectic structure Ω as

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.6}$$

Thus, if the phase space is a Riemannian manifold and therefore equipped with a metric g, the corresponding Kähler metric G is connected to g up to a factor that is determined by the choice (A.6) of the symplectic structure. It is straightforward to show that for a Euclidean space this coincides with the usual choice of a Euclidean metric $G = \mathbb{1}$.

While in the conventional context of Hamiltonian dynamics the metric is of no consequence, in the present paper we have a complex Hamiltonian function $\mathcal{H} = H - \mathrm{i}\Gamma$, and the metric structure appears as a natural choice to extend the canonical equations of motion to the generalized form

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \Omega^{-1}\vec{\nabla}H - G^{-1}\vec{\nabla}\Gamma, \tag{A.7}$$

where Ω is given by the usual expression (A.13) for Hamiltonian dynamics and G is the associated Kähler metric.

Let us now proceed to calculate the Kähler metric that is compatible with the choice (A.6) for the symplectic structure, in the case of the Bloch sphere. The Riemannian metric g on a two-sphere $S^2 \subset \mathbb{R}^3$ of radius R, embedded in the three-dimensional Euclidean phase space via

$$\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned} \tag{A.8}$$

can be found from the transformation rule (A.3) (which extends to the case in which we have a parametric subspace). Starting from the metric in $\mathbb{R}^3$ given by the identity matrix we deduce at once the well-known expression of the metric on the sphere:

$$g^{(\theta,\phi)} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}. \tag{A.9}$$

Because the sphere is the Bloch sphere, the radius is given by $R = \frac{1}{2}$. If we employ the usual coordinate system p and q according to

$$p = \cos\theta \qquad \text{and} \qquad q = \phi/2, \tag{A.10}$$

that is,

$$\theta = \arccos p \qquad \text{and} \qquad \phi = 2q, \tag{A.11}$$

then we find that the metric of our Bloch sphere is given by

$$g^{(p,q)} = \frac{1}{4}\begin{pmatrix} \frac{1}{1-p^2} & 0 \\ 0 & 4(1-p^2) \end{pmatrix}. \tag{A.12}$$

From this metric we can calculate the symplectic structure ω on the phase space (p, q) making use of the Kähler compatibility condition and find

$$\omega = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}. \tag{A.13}$$

As a consequence, by comparing (A.13) and (A.6) we see that the scaling factor κ is given by $\kappa = \frac{1}{2}$, that is, $\Omega = \frac{1}{2}\omega$, and we conclude that the Kähler metric associated with the symplectic structure appearing in the canonical equations of motion is given by $G = \frac{1}{2}g$.

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